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Causality Restrictions on Relativistic Extensions of Particle Symmetries†

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Abstract

Relativistic extensions of internal hadron symmetry groups are investigated from the viewpoint of causality requirements. Zeeman's group theoretical definition of causality is adopted and various physically interesting structures of relativistic extensions are studied from the viewpoint of whether they preserve or violate causality. Four theorems that guarantee causality preservation, and three theorems that violate it are deduced. It is concluded that there does not exist a non-trivial coupling of the Poincaré group and an internal symmetry group, such as $SU(3)$ or $SU(6)$, preserving causality in a Minkowski space. Extensions in complex or in curved manifolds are briefly discussed.

1. *Introduction*

Since the celebrated McGlinn theorem (McGlinn, 1964; Coester, *et al.*, 1964) of 1964, a large number of papers have been devoted to the coupling of the Poincaré group P with an internal hadron symmetry group S.

From a group theoretical viewpoint it has been proved that non-trivial couplings of P and S are certainly well-defined structures.

On the ground of physical applications however, the non-existence of mass splittings has been proved in many different frameworks (O'Raifeartaigh, 1965a, 1967a, b; Jost, 1966; Roman & Koh, 1965; Wegel, 1967; Coleman & Mandula, 1967). Although the 'no-go' theorems do not absolutely forbid a mass splitting, they are so strong that its existence practically requires some peculiar assumptions such as: infinite parameter Lie groups or algebras (Formanek, 1966); non-integrable or partially integrable representations (Doebner & Melsheimer, 1966); non-Lie algebras such as associative algebras (Bohm, 1967; Nakamura, 1967) or Lie-admissible algebras (Santilli, 1968b, 1968c).

As a complementary aspect of the above 'no-go' theorems, in the present paper we investigate the problem of the coupling of space-time and internal symmetries from a causality viewpoint.

In this connection a group theoretical definition of causality introduced by Zeeman (1964) is of particular interest. Zeeman introduces a partial

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ordering of a Minkowski space M if an event in $x \in M$ can influence an event in $y \in M$, according to

$$x < y$$

if and only if $(x-y)^2 = (x^0 - y^0)^2 - (\mathbf{x} - \mathbf{y})^2 > 0$ and $x^0 < y^0$.

Without any preliminary assumption of linearity or continuity, Zeeman proves that the largest group of automorphisms of the Minkowski space preserving the partial ordering, called the causality group C, is constituted by:

(i) the group D of dilatations in M;
(ii) the orthochronous Poincaré group $P\uparrow$;

and performs the map

$$x \to x' = \lambda \Lambda\uparrow x + a; \qquad \lambda \in D, (\Lambda\uparrow, a) \in P\uparrow, x \in M$$

Our problem essentially consists of investigating the action on the partial ordering of a larger group G containing as subgroup the Poincaré group P and an internal hadron symmetry group S.

In the following we always consider finite-parameter Lie groups and we assume for P a connected component of the orthochronous group $P\uparrow$; 'causality' always refers to Zeeman's causality in a $(3+1)$-dimensional Minkowski space M; finally, we denote by $\times$, $\otimes$ or $+$, $\oplus$, the direct and semidirect product, or the direct and semidirect sum, respectively.

2. *Extensions Preserving Causality*

We call a 'couple' $(G; G')$, where G' is a subgroup of a group G, a transitive couple if G acts transitively on the homogeneous space $H = G/G'$. In connection with our problem, we are interested in the couple of Lie groups $(P; L)$, where P is the connected Poincaré group and L its homogeneous Lorentz subgroup. $(P; L)$ is a transitive couple since, as is well known, P acts transitively on the Minkowski space $M = P/L$.

We call a 'triple' $(G; G', G'')$, where G' and G'' are subgroups of a Lie group G, a transitive decomposition of G if G' is transitive on the homogeneous space $H_1 = G/G''$ or G'' is transitive on $H_2 = G/G'$. Then (Oniščik, 1966) any element $g \in G$ can be given the form $g = g'g''$ with $g' \in G'$ and $g'' \in G''$. Conversely, if the imbeddings of two Lie groups G' and G'' in a larger Lie group G satisfies the assumption that any $g \in G$ can be given the form $g = g'g''$ with $g' \in G'$ and $g'' \in G''$, then the triple $(G; G', G'')$ constitutes a transitive decomposition.

The above property of the elements of transitive decompositions coincides with the minimality requirements (Greenberg, 1964; Michel, 1965) of the coupling of the connected Poincaré group P with an internal symmetry group S. Thus any triple $(G; P, S)$ satisfying the minimality requirement is a transitive decomposition.

We call a transitive decomposition $(G; G', G'')$ a semidirect decomposition if $G = G' \otimes G''$. A necessary and sufficient condition that a transitive decomposition $(G; G', G'')$ be a semidirect decomposition is that for one element $g' \in G'$, not contained in an invariant subgroup of G', $g', g'' g'^{-1} \in G''$ for all $g'' \in G''$ (Michel, 1965).

Theorem 1: Let $(G; P, S)$ be a transitive decomposition of a Lie group G with respect to the connected Poincaré group P and a connected semisimple group S. If S commutes with the homogeneous Lorentz group L and $P \cap S = \{\mathbf{1}\}$, then G preserves causality on the Minkowski space $M = P/L$.

Proof: We consider the coset space $H = G/L$ and we assume for elements of H the subsets of G of the right coset form

$$Lg = (lg \mid l \in L, g \in G)$$

Corresponding to a given $f \in G$ let

$$\rightarrow \sigma^f(Lg)$$

be a mapping of G into the permutation group of the set (Lg) of all right cosets of L with

$$\sigma^f(Lg) = (Lg \xrightarrow{\sigma^f} Lgf^{-1})$$

The mapping $f \rightarrow \sigma^f$ is a homomorphism, since $\sigma^f \sigma^{f'} = \sigma^{ff'}$ for any f, $f' \in G$. Since $(G; P, S)$ is a transitive decomposition, and by recalling that the decomposition $g = ps$, with $g \in G$, $p \in P$ and $s \in S$, is unique from $P \cap S = \{\mathbf{1}\}$ (Michel, 1965), we have $(Lg) = (Ps)$ and $\sigma^f(Lg) = \sigma^f(Ps)$. Consider now the mapping σ^l for $l \in L$. Since S commutes with L

$$\sigma^l(Ps) = (Psl^{-1}) = (P(sls^{-1})^{-1} s) = (Ps)$$

This implies that $P \subset \mathrm{Ker}(\sigma)$ since, from $sls^{-1} = l$, L is in $\mathrm{Ker}(\sigma)$ and $\mathrm{Ker}(\sigma) \cap P$ is an invariant subgroup of P. Consequently

$$\sigma^p(Ps) = (Ps)$$

for all $p \in P$, and P is an invariant subgroup of G. Finally, consider the automorphisms of P

$$m^s = (p' \mid p' = sps^{-1}; p \in P, s \in S)$$

The mapping $s \rightarrow m^s$ is a homomorphism of S into a $\mathrm{Aut}(P)$. Since S is semisimple and connected, it follows that (Michel, 1965; Hergfeldt & Hennig, 1969) $m^s = \mathbf{1}$, i.e. $sps^{-1} = p$. Thus S is mapped into the identity element of the center of $\mathrm{Aut}(P)$. But in the structure of $\mathrm{Aut}(P)$ the connected Poincaré group P is recovered by the invariant subgroup of inner automorphisms $\mathrm{Int}(P) \subset \mathrm{Aut}(P)$ (since P has no center), and the factor group $\mathrm{Aut}(P)/\mathrm{Int}(P)$ is isomorphic (modulo a cyclic group of two elements) to the group of dilatations D (Michel, 1965). Thus, under the mapping $s \rightarrow m^s$, S is mapped into the identity element of the subgroup of dilatations of the causality group, and G preserves causality.

Theorem 2: Let $(G; P, S)$ be a semidirect decomposition of a topological group G with respect to the connected Poincaré group P and a group S as topological subgroups. If S is compact, then G preserves causality on the Minkowski space $M = P/L$.

Proof: The group of all inner automorphisms $\mathrm{Int}(S)$ of a compact topological group S is a compact invariant subgroup of all continuous automorphisms $\mathrm{Aut}(S)$ of S (Iwasawa, 1949). Consequently, under the mapping

$$p \to m^p = (s' \mid s' = psp^{-1}; p \in P, s \in S)$$

P is imbedded into $\mathrm{Int}(S)$, since P is connected. Furthermore, the mapping $p \to m^p$ is the identity mapping since there does not exist a non-trivial continuous unitary finite-dimensional representation of P (Roskies, 1966). Conversely, the mapping

$$s \to m^s = (p' \mid p' = sps^{-1}; p \in P, s \in S)$$

is the identity mapping too, and a situation equivalent to the one of Theorem 1 occurs. Thus, structures of the type

$$G = T \otimes \{SU(n) \times L\}$$

preserves causality.

Theorem 3: Let $(G; P, S)$ be a transitive decomposition of a Lie group G with respect to the connected Poincaré group P and a group S. If S commutes with the homogeneous Lorentz group L and there does not exist an inner automorphism η of G mapping the one-parameter subgroup $p(t) \in P$ of temporal displacements according to $\eta p(t) \eta^{-1} = \alpha(\eta) p(-t)$, with $\alpha(\eta)$ a real number, then G preserves causality on the Minkowski space $M = P/L$.

Proof: Since S commutes with L by assumption, if S commutes with the translational subgroup T too, then causality is preserved as in Theorem 1. If S does not commute with T, then there exists a non-trivial real one-dimensional representation $\alpha(s)$ (i.e. S has a non-trivial abelian factor group) such that S acts on translations according to (Greenberg, 1964)

$$s\tau s^{-1} = \alpha(s)\,\tau$$

with $\alpha(s)$ a real number, for any $s \in S$ and $\tau \in T$. By recalling that space inversions preserve the partial ordering, the condition of non-existence of inner automorphisms of G performing the mapping of the one-parameter sub-group of temporal displacement $p(t)$ onto $p(-t)$ ensures the preservation of time ordering, and G preserves causality.

Theorem 4: Let $(G; P, S)$ be a semidirect decomposition of a Lie group G with respect to the connected Poincaré group P and a group S. If one of the following conditions holds:

(a) S is a semisimple group containing a subgroup isomorphic to the homogeneous Lorentz group L;

(b) there is at least one element p_0 in the Lie algrebra $\underset{\sim}{P}$ of P such that $[p_0, p] = [s, p]$ for any $p \in P$ and s in the Lie algebra $\underset{\sim}{S}$ of S;

(c) there is at least one element $s_0 \in \underset{\sim}{S}$ such that $[s_0, s] = [p, s]$ for any $p \in P$ and $s \in S$;

then there exists a redefinition $\hat{P}$ of P or $\hat{S}$ of S such that G preserves causality in $\hat{P}/\hat{L}$ or in P/L, respectively.

Proof: For case (a) there exists a redefinition $\hat{P}$ of P such that (Fleischman & Nagel, 1967; Roskies, 1966; Ottoson, *et al.*, 1965) the mapping

$$s \to m^s = (\hat{p}' \mid \hat{p}' = s\hat{p}s^{-1}; \hat{p} \in \hat{P}, s \in S)$$

is the identity mapping. Thus, as in Theorem 1, S is mapped into the center of $\mathrm{Aut}(\hat{P})$ and G preserves causality with respect to the redefined decomposition $(G; \hat{P}, S)$. If S is a semisimple group containing no subgroups isomorphic to L, then $\hat{P} = P$ (Fleischman & Nagel, 1967; Roskies, 1966) and G preserves causality without redefinition.

Corresponding to case (b), let us recall that the derivation algebra $\mathrm{Der}(\underset{\sim}{P})$ of $\underset{\sim}{P}$ is given by a semidirect sum of $\underset{\sim}{P}$ and a one-dimensional algebra $\underset{\sim}{D}$, i.e. $\mathrm{Der}(\underset{\sim}{P}) = \underset{\sim}{P} \oplus \underset{\sim}{D}$. Since $\underset{\sim}{P}$ has no center, a homomorphism of $\underset{\sim}{S}$ into $\mathrm{Der}(\underset{\sim}{P})$ is either a homomorphism into the algebra of inner derivation of $\underset{\sim}{P}$ or a homomorphism into the center of $\mathrm{Der}(\underset{\sim}{P})$. Condition (b) corresponds to the assumption that $\underset{\sim}{S}$ can be mapped into the inner derivation of $\underset{\sim}{P}$. But then there exists a redefinition $\hat{\underset{\sim}{S}}$ or $\underset{\sim}{S}$ such that $\underset{\sim}{G}$ is given by the direct sum $\underset{\sim}{G} = \underset{\sim}{P} + \hat{\underset{\sim}{S}}$ (Mugibayashi, 1966). This implies that $\hat{\underset{\sim}{S}}$ is mapped into the center of $\mathrm{Der}(\underset{\sim}{P})$. Then there exists a redefinition $\hat{S}$ of S such that the mapping

$$\hat{s} \to m^{\hat{s}} = (p' \mid p' = \hat{s}p\hat{s}^{-1}; p \in P, \hat{s} \subset \hat{S})$$

is the identity mapping, and G, as for Theorem 1, preserves causality with respect to the redefined decomposition $(G; P, \hat{S})$.

Similarly, if assumption (c) holds, on the basis of results equivalent to (b) with P and S interchanged (Hergfeldt & Hennig, 1969), there exists a redefinition $\hat{P}$ of P such that the mapping

$$s \to m^s = (\hat{p}' \mid \hat{p}' = s\hat{p}s^{-1}; \hat{p} \in \hat{P}, s \in S)$$

is the identity mapping and G preserves causality with respect to the redefined decomposition $(G; \hat{P}, S)$.

3. *Extensions Violating Causality*

In this section we derive three theorems which illustrate that the most interesting non-trivial extensions are bound to violate causality.

Theorem 5: Let $(G; P, S)$ be a semidirect decomposition of a connected Lie group G with respect to the connected Poincaré group P and a semisimple non-compact group S larger than the Lorentz group L and containing a subgroup isomorphic to it. If S does not commute with L, the G violates causality on the Minkowski space $M = P/L$.

Proof: Let us consider the coset space $Pg = (pg|p \in P, g \in G)$. Since $(G; P, S)$ is a semidirect decomposition, the set of all right cosets (Pg) of Pg is $(Pg) = (Ps|s \in S)$. Let, for $f \in G$, a mapping of G into the permutation group of (Ps) be given by

$$f \to \sigma^f(Ps) = (Psf^{-1})$$

Let $f = l \in L$. Since L does not commute with S, $\sigma^l(Ps) \neq (Ps)$. Thus P is not longer in $\mathrm{Ker}(\sigma)$ and S cannot be mapped into the identity element of the subgroup of dilatations of the causality group. Consequently, since S is larger than the Lorentz group and contains a subgroup isomorphic to it, violation of causality follows from the property that L is the largest semi-simple group preserving the partial ordering.

As an explicit example, let us consider the semidirect decomposition

$$G = P \otimes SU(3.1)$$

Let $W_{\mu\nu}$ (μ, $\nu = 0$, 1, 2, 3) be the Weyl basis of $\underline{SU}(3.1)$ with the usual decomposition in symmetric and antisymmetric components

$$S_{\mu\nu} = W_{\mu\nu} + W_{\nu\mu}, \qquad A_{\mu\nu} = W_{\mu\nu} - W_{\nu\mu}$$

where $A_{\mu\nu}$ span an algebra isomorphic to $\underline{L}$. Then violation of causality can be seen, for instance, by noting that there exist in G the set of elements induced by the symmetric generators $S_{\mu\nu}$ which do not preserve causality since they do not preserve the real form $(x-y)^2$ in the defining conditions of the partial ordering.

Theorem 6: Let G be a simple Lie group containing as subgroup the connected Poincaré group P. Then G violates causality on the Minkowski space $M = P/L$.

Proof: All possible transitive decompositions of complex simple Lie algebras $\underline{G}$ according to $(\underline{G}; \underline{G}', \underline{G}'')$ have been classified (Oniščik, 1966) and are summarized in Table 1, where $\underline{D}$ denotes a one-dimensional algebra. We now consider the real compact forms of the Lie algebras of the above classification and all the corresponding non-compact real forms obtained by means of inner involutive automorphisms (Gantmacher, 1939). By inspection we see that there does not exist a real non-compact form of $\underline{G}'$ or of $\underline{G}''$ isomorphic to the Lie algebra of the Poincaré group. Thus there does not exist a simple Lie group G admitting a transitive decomposition $(G; P, G'')$ or $(G; G', P)$ with respect to the connected Poincaré group P. Consequently, in all possible transitive decomposition $(G; G', G'')$ of (non-compact) simple Lie groups G, P is contained in a simple subgroup of G' or of G''. This proves the violation of causality since on one hand the 'closure' of $P = L \otimes T$ in a simple group requires supplementary non-abelian generators (as for the conformal group), and on the other hand P, modulo the abelian one-dimensional group of dilatations, is the largest group preserving the partial ordering.

TABLE 1

$\underline{G}$	$\underline{G}'$	$\underline{G}''$
$\underline{A}_{2n-1}$ $n > 1$	$\underline{C}_n$	$\underline{A}_{2n-2}$ $\underline{A}_{2n-2} \oplus \underline{D}$
$\underline{B}_3$	$\underline{G}_2$	$\underline{B}_2$ $\underline{B}_2 \oplus \underline{D}$ $\underline{D}_3$
$\underline{D}_{n+1}$ $n > 1$	$\underline{B}_n$	$\underline{A}_n$ $\underline{A}_n \oplus \underline{D}$
$\underline{D}_{2n}$	$\underline{B}_{2n-1}$	$\underline{C}_n$ $\underline{C}_n \oplus \underline{D}$ $\underline{C}_n \oplus \underline{A}_1$
$\underline{D}_8$	$\underline{B}_7$	$\underline{B}_4$ $\underline{B}_2$ $\underline{B}_2 \oplus \underline{D}$
$\underline{D}_4$	$\underline{B}_3$	$\underline{B}_2 \oplus \underline{A}_1$ $\underline{D}_3$ $\underline{D}_3 \oplus \underline{D}$ $\underline{B}_3$

For instance, for all the groups of the classes $SU(pq)$, $p \geqslant 2$, $q \geqslant 2$; $SO(p,q)$, $p \geqslant 4$, $q \geqslant 2$; $Sp(p,q)$, $p \geqslant 2$, $q \geqslant 2$; P is contained:

(i) as subgroup of $SU(2.2)$, corresponding for instance to $\underline{A}_3$ in a transitive decomposition connected to $(\underline{D}_4; \underline{B}_3, \underline{A}_3)$;
(ii) as subgroup of $SO(4.2)$, corresponding for instance to $\underline{D}_3$ in a transitive decomposition connected to $(\underline{B}_3; \underline{G}_2, \underline{D}_2)$;
(iii) as subgroup of any larger group containing $SU(2.2)$ or $SO(4.2)$.

But the conformal group $SO(4.2)$ does not preserve causality since, for instance, it violates the lemma 4 of the Zeeman paper (Zeeman, 1964) stating that each light ray in M is mapped linearly by any element of the causality group. By noting that $SO(4.2) = SU(2.2)/Z_2$ (Kihlberg, 1966), and by recalling that any G' or G'' of the transitive decomposition of the above groups, containing P a subgroup, must contain also as a subgroup the minimality group $SU(2.2)$ or $SO(4.2)$, the violation of causality follows for any group of the above classes.

In a more usual language, one can introduce as a minimality example the well known realization of the conformal algebra according to

$$\underline{SO}(4.2) = \underline{SU}(2.2) = (M_{\mu\nu}, P_\mu, D, A_\mu; \mu, \nu = 0, 1, 2, 3)$$

where $M_{\mu\nu}$, P_μ and D are the generators of the Lie algebra of the causality group and A_μ are the accelerations. Then there exist in the conformal group four elements induced by A_μ which do not preserve causality since the A_μ's are non-linear.

Theorem 7: Let G be a Lie group containing as subgroup the connected Poincaré group P. If the homogeneous Lorentz group L is imbedded in a larger simple subgroup of G, then G violates causality on the Minkowski space $M = P/L$.

Proof: Let us consider the Levi decomposition of the Lie algebra $\underline{G}$ of G

$$\underline{G} = \underline{R} \oplus \underline{F}$$

where $\underline{R}$ is the radical and $\underline{F}$ is the Levi factor, i.e. a semisimple Lie algebra. As is known (O'Raifeartaigh, 1965b), since $\underline{P} \subset \underline{G}$, the homogeneous Lorentz subalgebra $\underline{L}$ of $\underline{P}$ is contained in $\underline{F}$ in such a way that $[\underline{R}, \underline{L}] \subset \underline{R}$. Since Ad$\underline{L}$ acts irreducibly on the translational subalgebra $\underline{T}$ of $\underline{P}$, one has either $\underline{T} \cap \underline{R} = 0$, or $\underline{T} \cap \underline{R} = \underline{T}$. Thus the following cases are possible:

(a) $\underline{T} = \underline{R}$;
(b) $\underline{T} \cap \underline{R} = 0$;
(c) $\underline{T} \subset \underline{R}$, $\underline{R}$ abelian or non-abelian and solvable.

Let

$$\underline{G} = \underline{R} \oplus \left(\overset{k=n}{\underset{k=1}{+}} \underline{F}_k \right)$$

where $\underline{F}_k$'s are the totality of simple ideals in the decomposition of the Levi factor. Case (a) has been classified according to one of the two possibilities (Hergfeldt & Hennig, 1969)

(1) $\underline{G} = (\underline{T} \oplus \underline{L}) \overset{k=n}{\underset{k=2}{+}} \underline{F}_k$;

(2) $\underline{G} = (\underline{T} \oplus \underline{SL}(4, R)) \overset{k=n}{\underset{k=2}{+}} \underline{F}_k$.

Subcase (1) is excluded by the assumption of the Theorem. Subcase (2) violates causality, since $SL(4,R)$, as $SU(3.1)$ in the example of Theorem 5, does not preserve the defining conditions of the partial ordering. Case (b) reduces to Theorem 6, since it implies the imbedding of P in a simple algebra. Similarly the non-preservation of causality follows for case (c), since it implies the presence of a simple ideal in the decomposition of the Levi factor larger than and containing a subgroup isomorphic to L. As a particular example of case (c) one sees that the well-known relativistic extension of $SU(6)$

$$G = SL(6, c) \otimes T_{36}$$

violates causality.

4. *Concluding Remarks*

As we have seen, the preservation of causality practically restricts the coupling of the Poincaré group P with an internal hadron symmetry group S to trivial structures, such as the ones of Theorems 1, 2 and 3.

This result is so strong that it supports the idea of an enlargement of the basic homogeneous space $M = P/L$ according to an extended space

$$H = G/C$$

where $(G; C)$ is a transitive couple.

A first restriction for a minimality enlargement can be introduced by considering only transitive couples $(G; C)$ which are able to reproduce under contraction the basic couple $(P; L)$. An alternative, weaker restriction could be to consider an extension G of P which can reproduce under contraction the Poincaré group P. Supplementary restrictions could be introduced so as to admit physical interpretations, for instance, in connection with the existence of a one-parameter subgroup of temporal displacements.

In this way, as an example of enlargement, one gets the so-called 'physically admissible Lie groups' (Segal, 1967; Castell, 1968) which are connected Lie groups G such that

(A) a subgroup of G gives rise, via the Inönü-Wigner contraction, to the connected component of the Poincaré group;
(B) there exist in G a one-parameter subgroup $p(t)$ which can be interpreted as the subgroup of temporal displacements;
(C) there does not exist in G an inner automorphism η such that $\eta p(t)\eta^{-1} = \alpha(\eta)p(-t)$, with $\alpha(\eta)$ a real number.

The above enlargements allow the assumption, for instance, of the simple non-compact groups of the $SO(2,p)$ class, with $p = 3, 4, 5, \ldots$, which are contractible to $SO(1,p) \otimes T_{1+p}$ (Castell, 1968).

The most interesting aspect of the above approach is that now H can be irreducible Riemannian globally symmetric space (Helgason, 1962). Such extensions of particle symmetries on curved manifolds, for instance according to the DeSitter group (Roman & Aghassi, 1966), open up new possibilities of combining internal and space-time symmetries for which the results of the present paper have no relevance.

A second class of enlargements can be introduced without recursion to the Inönü-Wigner contraction, by performing a complex extension $\tilde{M}$ of the Minkowski space (Barut, 1964). In this connection it has been shown

(Santilli, 1968a) that a complex partial ordering can be introduced for any couple of points $z_1, z_2 \in \bar{M}$, with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, according to

$$z_1 <_c z_2$$

if $(x_1 - x_2)^2 > 0$, $x_1{}^0 < y_1{}^0$ and $(x_2 - y_2)^2 > 0$, $x_2{}^0 < y_2{}^0$. The group of causal automorphisms on $\bar{M}$ preserving the above partial ordering is then isomorphic, modulo a dilatation and the preservation of time orderings, to the group of transformations leaving invariant the quadratic form

$$(z_1 - z_2)(z_1 - z_2) \equiv (z_1{}^0 - z_2{}^0)^2 - (\mathbf{z}_1 - \mathbf{z}_2)^2$$

In this way one gets a complex causality group of orthogonal type given by Santilli (1968a)

$$\mathbf{C}_{\text{ort.}} = \mathbf{D} \otimes (\mathbf{T} \otimes \mathbf{L}\uparrow) = \mathbf{D} \otimes \mathbf{P}\uparrow$$

where $\mathbf{P}\uparrow$ is the 20-parameter inhomogeneous complex Lorentz group preserving time orientations.

A second extension of the Zeeman causality can be investigated by considering partial orderings in $\bar{M}$ expressed in terms of the real Hermitean form

$$(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \equiv |z_1{}^0 - z_2{}^0|^2 - |\mathbf{z}_1 - \mathbf{z}_2|^2$$

Conceivably, in this case one would get a complex causality group $\mathbf{C}_{\text{unit}}$ of unitary type expressible in terms of the inhomogeneous $SU(3.1)$ group

$$ISU(3.1) = \mathbf{T} \otimes SU(3.1)$$

which represents an interesting possibility of a non-trivial coupling of the Poincaré group and the $SU(3)$ group preserving causality in a complex space. Investigations along these lines are in progress.

Furthermore, the complex extension $\bar{M}$ of the Minkowski space M is physically supported by procedures such as the analytical continuation of Wightman functions (then a complex causality group of orthogonal type is directly involved), or the partial wave analysis of S-matrix elements in connection with Toller-like investigations of Regge daughter trajectories (Toller, 1968) (then both causality groups of orthogonal and of unitary type seem to be promising).

We thus conclude that there does not exist a non-trivial coupling of the Poincaré group and an internal hadron symmetry group, such as $SU(3)$ or $SU(6)$, preserving causality in Minkowski space. On the contrary, interesting possibilities for non-trivial links might exist for extensions in complex or in curved manifolds.

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